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ON THE INVESTIGATION OF OSCILLATIONS OF QUASILINEAR SYSTEMS WITH ALMOST-PERIODIC COEFFICIENTS

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A method is developed for investigating the oscillations of systems with almost-periodic coefficients, based on Kamenkov's ideas [1] on the construction of stationary solutions of systems with periodic coefficients and on the separation of motions. In contrast to [1] it is assumed that under the vanishing of a small parameter μ the system's characteristic equation has, besides n pairs of pure imaginary roots, m zero roots and h roots with negative real parts. Non-resonance and resonance cases are considered. Conditions are obtained for the existence of stationary solutions with respect to terms of first order in the small parameter. An example is presented.

1. We examine the problem of the existence and the structure of the solutions, stationary in the sense of [1], of a system whose motion obeys the equations (p_{ik} are constant coefficients and μ is a small positive parameter)

$$x_i^{\cdot} = \sum_{k=1}^{n_1} p_{ik} x_k + \sum_{j=1}^{\infty} \mu^j X_{ij}(x, t) + \sum_{j=0}^{\infty} \mu^j f_{ij}(t) \tag{1.1}$$

$$(x = x_1, \ldots, x_{n_1};\ i = 1, \ldots, n_1;\ n_1 = 2n + m + h)$$

Here the right hand sides are convergent series in parameter μ in the domain of variation being studied of the variables $x_1, \ldots, x_{n_1}$ and of the parameter: X_{ij} ($j = 1, 2, \ldots$) are polynomials of any finite order with coefficients almost-periodic in t, vanishing for $x_1 = \ldots = x_{n_1} = 0$; $f_{ij}(t)$ ($j = 0, 1, 2, \ldots$) are almost-periodic time functions. We prepresent the almost-periodic functions X_{ij} and f_{ij} by finite Fourier series with arbitrary frequency spectrum (generalized Fourier series). Any quasilinear system whose nonlinear terms have the structure of X_{ij}, while the coefficients of the linear terms are periodic functions of one and the same period, can be brought to form (1.1) [2]. Various cases wherein systems whose linear terms have almost-periodic coefficients are reducible to form (1.1) are known as well [3].

We assume that the characteristic equation $|p_{ik} - \delta_{ik}\rho| = 0$ has n pairs of pure imaginary roots, m zero roots, and h roots with negative real parts. The number of groups of solutions corresponding to the zero roots is assumed arbitrary. At first we assume that the condition

$$\sum_{s=1}^{n} k_s \lambda_s \neq E$$

for the absence of resonances is fulfilled for the pure imaginary roots $\pm i\lambda_s$ ($s = 1, \ldots, n$). Here k_s and E are positive and negative integers, including zero;

the k_s satisfy the condition

$$0 < \sum_s |k_s| \leqslant K$$

The number K is determined by the index of the highest form occurring in the X_{i1}. The eigenvalues λ_s are taken as being incommensurable with the frequency spectrum of the almost-periodic coefficients and functions occurring in the right hand sides of (1.1) (nonresonance case), i.e., relations analogous to those presented above are fulfilled for the frequencies λ_s and the frequency spectrum of the almost-periodic coefficients of functions X_{i1} and the functions f_{i0} and f_{i1}. The roots with negative real parts may be both simple as well as multiple with an arbitrary number of groups of solutions.

Under the assumptions made, system (1.1) is reduced to

$$y_s^{\cdot} = -\lambda_s z_s + \sum_{i=1}^{\infty} \mu^i Y_{si}(y, z, u, v, t) + \sum_{i=0}^{\infty} \mu^i \varphi_{si}(t) \qquad (1.2)$$

$$z_s^{\cdot} = \lambda_s y_s + \sum_{i=1}^{\infty} \mu^i Z_{si}(y, z, u, v, t) + \sum_{i=0}^{\infty} \mu^i \psi_{si}(t)$$

$$u_\alpha^{\cdot} = \sigma_{\alpha-1} u_{\alpha-1} + \sum_{i=1}^{\infty} \mu^i U_{\alpha i}(y, z, u, v, t) + \sum_{i=0}^{\infty} \mu^i \theta_{\alpha i}(t)$$

$$v_j^{\cdot} = \nu_j v_j + \varkappa_{j-1} v_{j-1} + \sum_{i=1}^{\infty} \mu^i V_{ji}(y, z, u, v, t) + \sum_{i=0}^{\infty} \mu^i \gamma_{ji}(t)$$

$$(y = y_1, \ldots, y_n;\ z = z_1, \ldots, z_n;\ u = u_1, \ldots, u_m;\ v = v_1, \ldots, v_h;\ s = 1, \ldots, n;\ \alpha = 1, \ldots, m;\ j = 1, \ldots, h;\ \sigma_0 = \varkappa_0 = 0)$$

by a nonsingular linear transformation [4]. Here y_s, z_s and u_α are critical variables, v_j are the variables of the adjoint system; the functions $Y_{si}, Z_{si}, U_{\alpha i}$ and Y_{ji} $(i = 1, 2, \ldots)$ have a structure analogous to that of X_{ij} and are polynomials of arbitrary finite degree in y, z, u and v, vanishing when $y = z = u = v = 0$, with coefficients almost-periodic in t. The number different from zero $\sigma_{\alpha-1}$ and $\varkappa_{j-1}$ is determined by the multiplicity of the zero roots and of the roots with negative real parts and by the number of groups of solutions corresponding to these roots.

To solve the problem posed we show that nonautonomous transformations with almost-periodic coefficients exist leading (1.2) to a form when:

functions corresponding to $\varphi_{si}(t)$, $\psi_{si}(t)$, $\theta_{\alpha i}(t)$ and $\gamma_{ji}(t)$ $(i = 0, 1)$ in the original system are absent in the transformed system;

terms of first order in μ, depending on the noncritical variables $v_1, \ldots, v_h$ are absent in the critical system;

in the critical system, up to μ to the first power, inclusive, the coefficients of polynomials are independent of time.

From the system thus transformed we obtain the equations for the stationary amplitudes and we construct the desired stationary solutions. By investigating the approximate solutions found for stability and by estimating the magnitudes of their deviations from the solutions of the complete system, we determine the conditions for the existence of stationary oscillations from the first-order terms in parameter μ.

1°. Setting $\xi_s{}^* = y_s + iz_s$ and $\bar{\xi}_s{}^* = y_s - iz_s$ in (1.2), we pass to the complex form of writing the subsystem with pure imaginary roots and we transform the resulting system by the changes

$$\xi_s{}^* = \xi_s + W_{s0}(t) + \mu W_{s1}(t), \quad u_\alpha = \eta_\alpha + \omega_{\alpha 0}(t) + \mu\omega_{\alpha 1}(t) \qquad (1.3)$$
$$v_j = \zeta_j + \chi_{j0}(t) + \mu\chi_{j1}(t)$$

Here W_{s0}, W_{s1}, $\omega_{\alpha 0}$, $\omega_{\alpha 1}$, χ_{j0} and χ_{j1} are defined such that the functions corresponding to $\varphi_{s0}{}^*$, $\varphi_{s1}{}^*$, $\theta_{\alpha 0}$, $\theta_{\alpha 1}$, γ_{j0} and γ_{j1} of the original system vanish in the transformed system. For this it is sufficient that W_{s0}, W_{s1}, $\omega_{\alpha 0}$, $\omega_{\alpha 1}$, χ_{j0} and χ_{j1} satisfy the equations

$$\begin{aligned}
&dW_{s0}/dt = i\lambda_s W_{s0} + \varphi_{s0}{}^*(t) \qquad (1.4)\\
&dW_{s1}/dt = i\lambda_s W_{s1} + \varphi_{s1}{}^*(t) + \Xi_{s1}{}^*(W_0, \overline{W}_0, \omega_0, \chi_0, t)\\
&d\omega_{\alpha 0}/dt = \sigma_{\alpha-1}\omega_{\alpha-1,0} + \theta_{\alpha 0}(t)\\
&d\omega_{\alpha 1}/dt = \sigma_{\alpha-1}\omega_{\alpha-1,1} + \theta_{\alpha 1}(t) + U_{\alpha 1}(W_0, \overline{W}_0, \omega_0, \chi_0, t)\\
&d\chi_{j0}/dt = \nu_j\chi_{j0} + \varkappa_{j-1}\chi_{j-1,0} + \gamma_{j0}(t)\\
&d\chi_{j1}/dt = \nu_j\chi_{j1} + \varkappa_{j-1}\chi_{j-1,1} + \gamma_{j1}(t) + V_{j1}\\
&(W_0 = W_{10}, \ldots, W_{n0};\ \overline{W}_0 = \overline{W}_{10}, \ldots, \overline{W}_{n0};\ \omega_0 = \omega_{10}, \ldots, \omega_{m0};\ \chi_0 = \chi_{10}, \ldots, \chi_{h0};\ \varphi_{sk}{}^* = \varphi_{sk} + i\psi_{sk}\ (k = 0, 1);\\
&\Xi_{s1}{}^* = Y_{s1} + iZ_{s1})
\end{aligned}$$

The first four of Eqs. (1.4) have almost-periodic solutions if and only if

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-i\lambda_s t}\varphi_{s0}{}^*(t)\,dt = \qquad (1.5)$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-i\lambda_s t}[\varphi_{s1}{}^*(t) + \Xi_{s1}{}^*(W_0, \overline{W}_0, \chi_0, t)]\,dt = 0$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \theta_{\alpha 0}(t)\,dt = \lim_{t\to\infty}\frac{1}{t}\int_0^t [\theta_{\alpha 1}(t) + U_{\alpha 1}(W_0, \overline{W}_0, \omega_0, \chi_0, t)]\,dt = 0$$

The first condition in (1.5) is always fulfilled under the assumptions made on the incommensurability of λ_s with the frequency spectrum of the almost-periodic functions. In what follows we assume the fulfilment of the second condition in (1.5).

Since $\operatorname{Re}\nu_j \neq 0$, by virtue of the Neugebauer — Bohr theorem [3,5] the last two of Eqs. (1.4) always have almost-periodic solutions when $\gamma_{j0}(t)$ and $\gamma_{j1}(t)$ are almost-periodic. As a result of transformation (1.3) the system becomes

$$\xi_s^{\cdot} = i\lambda_s\xi_s + \sum_{i=1}^{\infty}\mu^i\Xi_{si}(\xi, \bar{\xi}, \eta, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\varphi_{si}{}^*(t) \qquad (1.6)$$
$$\bar{\xi}_s^{\cdot} = -i\lambda_s\bar{\xi}_s + \sum_{i=1}^{\infty}\mu^i\overline{\Xi}_{si}(\xi, \bar{\xi}, \eta, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\overline{\varphi}_{si}{}^*(t)$$
$$\eta_\alpha^{\cdot} = \sigma_{\alpha-1}\eta_{\alpha-1} + \sum_{i=1}^{\infty}\mu^i H_{\alpha i}(\xi, \bar{\xi}, \eta, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\theta_{\alpha i}{}^*(t)$$
$$\zeta_j^{\cdot} = \nu_j\zeta_j + \varkappa_{j-1}\zeta_{j-1} + \sum_{i=1}^{\infty}\mu^i G_{ji}(\xi, \bar{\xi}, \eta, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\gamma_{ji}{}^*(t)$$

$$(\xi = \xi_1, \ldots, \xi_n;\ \eta = \eta_1, \ldots, \eta_m;\ \zeta = \zeta_1, \ldots, \zeta_h)$$

2°. We represent Ξ_{s1} and $H_{\alpha 1}$ in the form

$$\Xi_{s1} = \Xi_{s1}^{(0)}(\xi, \bar{\xi}, \eta, t) + \sum_{l \geqslant 0} E_{s1}^{(l)}(\zeta, t)\, \xi_1^{l_1} \ldots \xi_n^{l_n} \bar{\xi}_1^{l_{n+1}} \ldots \bar{\xi}_n^{l_{2n}} \eta_1^{l_{2n+1}} \ldots \eta_m^{l_{2n+m}}$$

$$H_{\alpha 1} = H_{\alpha 1}^{(0)}(\xi, \bar{\xi}, \eta, t) + \sum_{l \geqslant 0} F_{\alpha 1}^{(l)}(\zeta, t)\, \xi_1^{l_1} \ldots \xi_n^{l_n} \bar{\xi}_1^{l_{n+1}} \ldots \bar{\xi}_n^{l_{2n}} \eta_1^{l_{2n+1}} \ldots \eta_m^{l_{2n+m}}$$

$$E_{s1}^{(l)}(\zeta, t) = \sum_{k \geqslant 1} e_{s1(l)}^{(k)}(t)\, \zeta_1^{k_1} \ldots \zeta_h^{k_h}$$

$$F_{\alpha 1}^{(l)}(\zeta, t) = \sum_{k \geqslant 1} f_{\alpha 1(l)}^{(k)}(t)\, \zeta_1^{k_1} \ldots \zeta_h^{k_h}$$

$$(l) = (l_1, \ldots, l_{2n+m});\ l = l_1 + \ldots + l_{2n+m}$$

$$(k) = (k_1, \ldots, k_h);\ k = k_1 + \ldots + k_h$$

Here $e_{s1(l)}^{(k)}(t)$ and $f_{\alpha 1(l)}^{(k)}(t)$ are known almost-periodic functions of t. We introduce the change of variables

$$\xi_s = p_s + \mu \sum_{l \geqslant 0} \sum_{k \geqslant 1} g_{s1(l)}^{(k)} \zeta_1^{k_1} \ldots \zeta_h^{k_h} \xi_1^{l_1} \ldots \eta_m^{l_{2n+m}} \tag{1.7}$$

$$\eta_\alpha = q_\alpha + \mu \sum_{l \geqslant 0} \sum_{k \geqslant 1} h_{\alpha 1(l)}^{(k)} \zeta_1^{k_1} \ldots \zeta_h^{k_h} \xi_1^{l_1} \ldots \eta_m^{l_{2n+m}}$$

and we define $g_{s1(l)}^{(k)}$ and $h_{\alpha 1(l)}^{(k)}$ such that in the transformed system the polynomials corresponding to $E_{s1}^{(l)}(\zeta, t)$ and $F_{\alpha 1}^{(l)}(\zeta, t)$ vanish. Then, starting from the functions

$$g_{s1(0,\ldots,0,l)}^{(0,\ldots,0,k)} \quad \text{and} \quad h_{\alpha 1(0,\ldots,0,l)}^{(0,\ldots,0,k)}$$

for prescribed l and k and, next, defining

$$g_{s1(0,\ldots,0,l)}^{(0,\ldots,1,k-1)} \quad \text{and} \quad h_{\alpha 1(0,\ldots,0,l)}^{(0,\ldots,1,k-1)}$$

etc. [6], for $g_{s1(l)}^{(k)}$ and $h_{\alpha 1(l)}^{(k)}$ we obtain the equations

$$dg_{s1(l)}^{(k)}/dt = (i\delta_1 + \delta)\, g_{s1(l)}^{(k)} + c_{s1(l)}^{(k)}(t) \tag{1.8}$$

$$dh_{\alpha 1(l)}^{(k)}/dt = (i\delta_2 + \delta)\, h_{\alpha 1(l)}^{(k)} + d_{\alpha 1(l)}^{(k)}(t)$$

$$\delta_2 = \sum_{j=1}^{n} (l_{n+j} - l_j)\lambda_j; \quad \delta_1 = \delta_2 + \lambda_s; \quad \delta = -\sum_{i=1}^{h} k_i \nu_i$$

Here $c_{s1(l)}^{(k)}(t)$ and $d_{\alpha 1(l)}^{(k)}(t)$ are known almost-periodic functions that are combinations of $e_{s1(l)}^{(k)}(t)$, $f_{\alpha 1(l)}^{(k)}(t)$ and of the functions $g_{s1(l)}^{(k)}(t)$, $h_{\alpha 1(l)}^{(k)}(t)$ found earlier. In the general case the expressions $i\delta_1 + \delta$ and $i\delta_2 + \delta$ have nonzero real parts. Consequently, each of Eqs. (1.8) has one and only one almost-periodic solution [5]. After changes (1.7) system (1.6) takes the form

$$p_s^{\cdot} = i\lambda_s p_s + \mu P_{s1}(p, \bar{p}, q, t) + \tag{1.9}$$

$$\sum_{i=2}^{\infty} \mu^i P_{si}(p, \bar{p}, q, \zeta, t) + \sum_{i=2}^{\infty} \mu^i \varphi_{si}^*(t)$$

$$\bar{p}_s^{\cdot} = -i\lambda_s\bar{p}_s + \mu\bar{P}_{s1}(p, \bar{p}, q, t) + \sum_{i=2}^{\infty}\mu^i\bar{P}_{si}(p, \bar{p}, q, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\bar{\varphi}_{si}^*(t)$$

$$q_\alpha^{\cdot} = \sigma_{\alpha-1}q_{\alpha-1} + \mu Q_{\alpha 1}(p, \bar{p}, q, t) + \sum_{i=2}^{\infty}\mu^i Q_{\alpha i}(p, \bar{p}, q, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\theta_{\alpha i}^*(t)$$

$$\zeta_j^{\cdot} = \nu_j\zeta_j + \varkappa_{j-1}\zeta_{j-1} + \sum_{i=1}^{\infty}\mu^i G_{ji}^*(p, \bar{p}, q, \zeta, t) + \sum_{i=2}^{\infty}\mu^i\gamma_{ji}^*(t)$$

Here the polynomials P_{si}, $\bar{P}_{si}$, $Q_{\alpha i}$ and G_{ji}^* $(i = 1, 2, \ldots)$ retain the structure of functions X_{ij}.

3°. We transform system (1.9) be setting [7]

$$p_s = p_s^* + \mu\sum_{l\geqslant 1} a_{s1}^{(l)}(t)\, p_1^{l_1}\ldots p_n^{l_n}\bar{p}_1^{l_{n+1}}\ldots\bar{p}_n^{l_{2n}}q_1^{l_{2n+1}}\ldots q_m^{l_{2n+m}} \qquad (1.10)$$

$$q_\alpha = \rho_\alpha + \mu\sum_{l\geqslant 1} b_{\alpha 1}^{(l)}(t)\, p_1^{l_1}\ldots p_n^{l_n}\bar{p}_1^{l_{n+1}}\ldots\bar{p}_n^{l_{2n}}q_1^{l_{2n+1}}\ldots q_m^{l_{2n+m}}$$

We choose the functions $a_{s1}^{(l)}$ and $b_{\alpha 1}^{(l)}$ in such a way that the polynomials playing the role of P_{s1} and $Q_{\alpha 1}$ in the transformed system do not depend on time. Then $a_{s1}^{(l)}$ and $b_{\alpha 1}^{(l)}$ must satisfy the equations (once again we define the functions in the above-mentioned order)

$$\frac{da_{s1}^{(l)}}{dt} + i\delta_1 a_{s1}^{(l)} = M_{s1}^{(l)}(t) - N_{s1}^{*(l)}, \quad \frac{db_{\alpha 1}^{(l)}}{dt} + i\delta_2 b_{\alpha 1}^{(l)} = L_{\alpha 1}^{(l)}(t) - K_{\alpha 1}^{(l)} \qquad (1.11)$$

Here $M_{s1}^{(l)}(t)$ and $L_{\alpha 1}^{(l)}(t)$ are known time functions, being collections of coefficients of polynomials P_{s1}, $\bar{P}_{s1}$, $Q_{\alpha 1}$ and of functions $a_{s1}^{(l)}$, $b_{\alpha 1}^{(l)}$; $N_{s1}^{*(l)}$ and $K_{\alpha 1}^{(l)}$ are the coefficients, yet to be determined, of the polynomials in the right hand sides of the system resulting after transformation (1.10).

We seek almost-periodic solutions of Eqs. (1.11). Two cases are possible:

δ_1 and δ_2 vanish; then the equations have almost-periodic solutions if $N_{s1}^{*(l)}$ and $K_{\alpha 1}^{(l)}$ have been determined by the equalities

$$N_{s1}^{*(l)} = \lim_{t\to\infty}\frac{1}{t}\int_0^t M_{s1}^{(l)}(t)\,dt, \quad K_{\alpha 1}^{(l)} = \lim_{t\to\infty}\frac{1}{t}\int_0^t L_{\alpha 1}^{(l)}(t)\,dt$$

Consequently, $N_{s1}^{*(l)}$ and $K_{\alpha 1}^{(l)}$ are constants (zero, in a special case);

δ_1 and δ_2 are nonzero. We represent the functions $M_{s1}^{(l)}$ and $L_{\alpha 1}^{(l)}$ as generalized Fourier series

$$M_{s1}^{(l)} = \sum_{k=1}^{N} M_{s1,\,k}^{(l)} e^{im_k t}, \quad L_{\alpha 1}^{(l)} = \sum_{k=0}^{N} L_{\alpha 1,\,k}^{(l)} e^{im_k t}$$

Here $M_{s1,k}^{(l)}$ and $L_{\alpha 1,k}^{(l)}$ are complex constants and m_k $(k = 0, 1, \ldots, N)$ are arbitrary real numbers. In this case the particular solutions of system (1.11) take the form

$$a_{s1}^{(l)} = e^{-i\delta_1 t} \sum_{k=0}^{N} \int_0^t e^{i(\delta_1+m_k)t} M_{s1,k}^{(l)} dt + \frac{i}{\delta_1} (1 - e^{-i\delta_1 t}) N_{s1}^{*(l)}$$

$$b_{\alpha 1}^{(l)} = e^{-i\delta_2 t} \sum_{k=0}^{N} \int_0^t e^{i(\delta_2+m_k)t} L_{\alpha 1,k}^{(l)} dt + \frac{i}{\delta_2} (1 - e^{-i\delta_2 t}) K_{\alpha 1}^{(l)}$$

and are almost-periodic for any $N_{s1}^{*(l)}$ and $K_{\alpha 1}^{(l)}$ by virtue of the fact that $\delta_1 + m_k \neq 0$ and $\delta_2 + m_k \neq 0$ $(k = 0, \ldots, N)$ (nonresonance case). We set $N_{s1}^{*(l)} = K_{\alpha 1}^{(l)} = 0$. As a result of the transformations the system becomes

$$\begin{aligned}
&p_s^{*\cdot} = i\lambda_s p_s^* + \mu P_{s1}^* (p^*, \bar{p}^*, \rho) + O(\mu^2) \\
&\bar{p}_s^{*\cdot} = -i\lambda_s \bar{p}_s^* + \mu \bar{P}_{s1}^* (p^*, \bar{p}^*, \rho) + O(\mu^2) \\
&\rho_\alpha^{\cdot} = \sigma_{\alpha-1}\rho_{\alpha-1} + \mu D_{\alpha 1} (p^*, \bar{p}^*, \rho) + O(\mu^2) \\
&\zeta_j^{\cdot} = \nu_j \zeta_j + \varkappa_{j-1}\zeta_{j-1} + \mu G_{j1} (p^*, \bar{p}^*, \rho, \zeta, t) + O(\mu^2) \\
&P_{s1}^* = p_s^* \sum_{l \geqslant 0} N_{s1}^{*(l)} (p_1^* \bar{p}_1^*)^{l_1} \ldots (p_n^* \bar{p}_n^*)^{l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} \\
&D_{\alpha 1} = \sum_{l \geqslant 1} K_{\alpha 1}^{(l)} (p_1^* \bar{p}_1^*)^{l_1} \ldots (p_n^* \bar{p}_n^*)^{l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} \\
&(l) \equiv (l_1, \ldots, l_n, l_1, \ldots, l_n, l_{2n+1}, \ldots, l_{2n+m}) \\
&l = 2l_1 + \ldots + 2l_n + l_{2n+1} + \ldots + l_{2n+m}
\end{aligned} \tag{1.12}$$

We observe that any one of the two equations in complex form is equivalent to two equations for pure imaginary roots in canonic form. If in (1.12) we pass to the polar coordinates $p_s^* = r_s e^{i\theta_s}$ and $\bar{p}_s^* = r_s \bar{e}^{i\theta_s}$, we obtain

$$\begin{aligned}
&r_s^{\cdot} = \mu r_s \sum_{l \geqslant 0} A_{s1}^{(l)} r_1^{2l_1} \ldots r_n^{2l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} + O(\mu^2) \\
&\theta_s^{\cdot} = \lambda_s + \mu \sum_{l \geqslant 0} B_{s1}^{(l)} r_1^{2l_1} \ldots r_n^{2l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} + O(\mu^2) \\
&\rho_\alpha^{\cdot} = \sigma_{\alpha-1}\rho_{\alpha-1} + \mu \sum_{l \geqslant 1} C_{\alpha 1}^{(l)} r_1^{2l_1} \ldots r_n^{2l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} + O(\mu^2) \\
&\zeta_j^{\cdot} = \nu_j \zeta_j + \varkappa_{j-1}\zeta_{j-1} + \mu G_{j1} (r, \theta, \rho, \zeta, t) + O(\mu^2) \\
&(r = r_1, \ldots, r_n;\ \theta = \theta_1, \ldots, \theta_n;\ \rho = \rho_1, \ldots, \rho_m)
\end{aligned} \tag{1.13}$$

Here $A_{s1}^{(l)}$ and $B_{s1}^{(l)}$ are, respectively, the real and the imaginary parts of $N_{s1}^{*(l)}$.

Let us consider the equations

$$\begin{aligned}
&r_s \sum_{l \geqslant 0} A_{s1}^{(l)} r_1^{2l_1} \ldots r_n^{2l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} = 0 \\
&\sigma_{\alpha-1}\rho_{\alpha-1} + \mu \sum_{l \geqslant 1} C_{\alpha 1}^{(l)} r_1^{2l_1} \ldots r_n^{2l_n} \rho_1^{l_{2n+1}} \ldots \rho_m^{l_{2n+m}} = 0
\end{aligned} \tag{1.14}$$

If system (1.14) has real nonnegative solutions $r_{10}^2, \ldots, r_{n0}^2$ and real solutions $\rho_{10}, \ldots, \rho_{m0}$, then, by substituting them into the transformations taking (1.1) to (1.13), we obtain the desired stationary solutions to within first order in μ. The solutions of the second equation in system (1.13) for $\theta_1, \ldots, \theta_n$ and the bounded, in particular, almost-periodic solutions for $\zeta_1, \ldots, \zeta_h$ are determined, to within first order in μ, from the equations

$$\theta_s^{\cdot} = \lambda_s + \mu \sum_{l\geqslant 0} B_{s1}^{(l)} r_{10}^{2l_1} \ldots r_{n0}^{2l_n} \rho_{10}^{l_{2n+1}} \ldots \rho_{m0}^{l_{2n+m}} \qquad (1.15)$$
$$\zeta_j^{\cdot} = \nu_j \zeta_j + \varkappa_{j-1}\zeta_{j-1} + \mu G_{j1}(r_0, \theta_0, \rho_0, \zeta, t)$$

Here $\theta_0 = \theta_{10}, \ldots, \theta_{n0}$ are solutions of the equations for θ_s.

Let us investigate the existence and the accuracy of the solutions obtained [1, 8]. Let $r_s = r_{s0} + x_s^*$, $\rho_\alpha = \rho_{\alpha 0} + x_{n+\alpha}^*$ and $\zeta_j = \zeta_{j0} + \zeta_j^*$, where $\zeta_{j0} = \zeta_{j0}(t)$ are bounded solutions of the second group of equations in (1.15). Then, by virtue of (1.13) the deviations x_i^* $(i = 1, \ldots, n+m)$ and ζ_j^* satisfy the equations

$$x_i^{*\cdot} = \sigma_{i-1}x_{i-1}^* + \mu \sum_{k=1}^{n+m} a_{ik}x_k^* + \mu X_{i1}^*(x^*) + \mu^2 X_{i2}^*(x^*, z^*, \theta t, \mu) \qquad (1.16)$$
$$z_j^{*\cdot} = \sum_{l=1}^{h} b_{jl}z_l^* + \mu Z_{j1}^*(x^*, z^*, \theta, t) + \mu^2 Z_{j2}^*(x^*, z^*, \theta, t, \mu)$$
$$(x = x_1^*, \ldots, x_{n+m}^*;\ z^* = z_1^*, \ldots, z_h^*;\ \theta = \theta_1, \ldots, \theta_n;\ i = 1, \ldots, n+m;\ j = 1, \ldots, h)$$

Here X_{i1} and Z_{j1} do not contain forms of lower than second and first order, respectively; some of the numbers σ_{i-1} are zero.

Let the roots of the characteristic equation

$$|\, \mu a_{ik} + \delta_{i-1,k-1}'\sigma_{i-1} - \delta_{ik}\lambda_0 \,| = 0 \qquad (1.17)$$

where $\delta_{i-1,k-1}' = 0$ for $i \neq k$ and $\delta_{i-1,k-1}' = 1$, for $i = k$, have only negative real parts. If in the right hand sides of the system we discard terms with powers of μ higher than first the solutions of the resulting system differ by an arbitrarily small amount from the solutions of the complete system (1.13) with a sufficiently small μ. Let us prove this, assuming that all roots of the characteristic equation are simple and real. From the Newton diagram [9] it follows that we can represent all roots of (1.17) in the form $\lambda_{0i} = \mu^{\varepsilon_i}h_{0i}$, where $0 < \varepsilon_i \leqslant 1$. Let all the roots be negative and distinct. Then, retaining the previous notation, by a linear change we transform system (1.16) to the form

$$x_i^{*\cdot} = \lambda_{0i}x_i^* + \mu X_{i1}^*(x^*) + \mu^2 X_{i2}^*(x^*, z^*, \theta, t, \mu) \qquad (1.18)$$
$$z_j^{*\cdot} = \sum_l b_{jl}z_l^* + \mu Z_{j1}^*(z^*, x^*, \theta, t) + \mu^2 Z_{j2}^*(x^*, z^*, \theta, t, \mu)$$

We choose the Liapunov function

$$V = \frac{1}{2}\sum_i x_i^{*2} + V_1(z^*)$$

Here $V_1(z^*)$ is determined by the equation

$$\sum_{j=1}^{h} \frac{\partial V_1}{\partial z_j^*} \sum_{l=1}^{h} b_{jl}z_l^* = -\sum_{j=1}^{h} z_j^{*2}$$

Function $V_1(z^*)$ is sign-definite because $\operatorname{Re}\nu_j < 0$. Setting $x_i^* = r^*\cos\gamma_i^*$, $z_j^* = r^*\cos\gamma_{n+m+j}^*$ and computing the derivative of function V relative to (1.18), we obtain

$$V^{\cdot} = r^{*2}\sum_i \mu^{\varepsilon_i} h_{0i}\cos^2\gamma_i^* - r^{*2}\sum_j \cos^2\gamma_{n+m+j}^* + \sum_{k=1}^{4}\Lambda_k$$

$$\Lambda_1 = \mu r^{*3} H_1(\gamma_1^*, \ldots, \gamma_{n+m}^*, r^*)$$

$$\Lambda_2 = \mu^2 r^* H_2(\gamma_1^*, \ldots, \gamma_{n+m+h}^*, r^*, t, \theta, \mu)$$

$$\Lambda_3 = \mu r^{*2} H_3(\gamma_1^*, \ldots, \gamma_{n+m+h}^*, r^*, \theta, t)$$

$$\Lambda_4 = \mu^2 r^* H_4(\gamma_1^*, \ldots, \gamma_{n+m+h}^*, r^*, \theta, t, \mu)$$

Here $0 < \varepsilon_i \leqslant 1$, $h_{0i} < 0$, the functions $\mu^n r^{*m} H_k$ are various products of $\mu^n x_i^* X_{ij}$ and of the functions

$$\Sigma \partial V_1 / \partial z^*_j \mu^n Z_{ij}^*$$

For a sufficiently small μ, $V^{\cdot} < 0$ on the sphere

$$\sum_i x_i^{*2} + \sum_j z_j^{*2} = r^{*2} = (\mu^{1-\varepsilon})^2 \qquad (1.19)$$

where ε is an arbitrarily small positive number. Consequently, if the initial values of x_i^* and z_j^* satisfy the conditions $|x_i^*(t_0)| < L$ and $|z_j^*(t_0)| < L$ (L defines a domain into which the points of the surface $V = M$ do not penetrate, where M is the lower bound of V on sphere (1.19)), then $|x_i^*(t)| < \mu^{1-\varepsilon}$ and $|z_j^*(t)| < \mu^{1-\varepsilon}$ for any t in the interval $(0, \infty)$.

Analogous arguments are valid for the case of multiple and complex roots. If among the λ_{0i} even one has a positive real part, the resulting solutions will not arbitrarily slightly differ from the solutions of the complete system. If among the λ_{0i} even one has a zero real part, then the problem of the existence of stationary oscillations in system (1.1) cannot be solved by using just terms of first order in μ.

2. Let us consider the resonance case. In contrast to Sect. 1 we assume that resonances of the form

$$\sum_s \lambda_s k_s = E$$

are possible for the pure imaginary roots $\pm i\lambda_s$ ($s = 1, \ldots, n$) of the characteristic equation of system (1.1). We take it that the λ_s are commensurable with the frequency spectrum of the almost-periodic coefficients and functions in (1.1). The pure imaginary roots may be multiple with an arbitrary number of groups of solutions. In addition, we assume that when $X_{i1} \equiv 0$ system (1.1) admits of almost-periodic solutions correct to within first order in μ. For the resonance case the system's canonic form is

$$y_s^{\cdot} = -\lambda_s z_s + \beta_{s-1} y_{s-1} + \sum_{i=1}^{\infty}\mu^i Y_{si}(y, z, u, v, t) + \sum_{i=0}^{\infty}\mu^i \varphi_{si}(t) \qquad (2.1)$$

$$z_s^{\cdot} = \lambda_s y_s + \beta_{s-1} z_{s-1} + \sum_{i=1}^{\infty}\mu^i Z_{si}(y, z, u, v, t) + \sum_{i=0}^{\infty}\mu^i \psi_{si}(t)$$

$$u_\alpha^{\cdot} = \sigma_{\alpha-1} u_{\alpha-1} + \sum_{i=1}^{\infty}\mu^i U_{\alpha i}(y, z, u, v, t) + \sum_{i=0}^{\infty}\mu^i \theta_{\alpha i}(t)$$

$$v_j^{\cdot} = \nu_j v_j + \varkappa_{j-1} v_{j-1} + \sum_{i=1}^{\infty} \mu^i V_{ji}(y, z, u, v, t) + \sum_{i=0}^{\infty} \mu^i \gamma_{ji}(t)$$
$$(\beta_0 = \sigma_0 = \varkappa_0 = 0)$$

The general method of investigation of the system is analogous to that in Sect. 1 (also see [10]).

We apply a transformation analogous to that in 1° of Sect. 1 for the nonresonance case. After passing from variables y_s and z_s to the complex-conjugate variables ξ_s and $\bar{\xi}_s$ and after the changes

$$\xi_s = p_s e^{i\lambda_s t}, \ \bar{\xi}_s = \bar{p}_s e^{-i\lambda_s t} \quad (s = 1, \ldots, n)$$

and taking into account that $\lambda_s = \lambda_{s-1}$ if $\beta_{s-1} \neq 0$, we obtain

$$p_s^{\cdot} = \beta_{s-1} p_{s-1} + \mu P_{s1} + O(\mu^2), \quad \bar{p}^{\cdot}_s = \beta_{s-1} \bar{p}_{s-1} + \mu \bar{P}_{s1} + O(\mu^2) \tag{2.2}$$
$$\eta_\alpha^{\cdot} = \sigma_{\alpha-1} \eta_{\alpha-1} + \mu H_{\alpha 1}^* + O(\mu^2), \quad \zeta_j^{\cdot} = \nu_j \zeta_j + \varkappa_{j-1} \zeta_{j-1} + \mu G_{j1} + O(\mu^2)$$

Here the number of nonzero β_{s-1} is determined by the multiplicity of the critical roots and by the number of groups of solutions corresponding to the roots mentioned. The transformations in 2° and 3° are carried out analogously to the nonresonance case and system (2.2) is reduced to

$$r_i^{\cdot} = g_{i-1} r_{i-1} + \mu \sum_{l \geqslant 1} R_{i1}^{(l)} r_1^{l_1} \ldots r_{2n+m}^{l_{2n+m}} + O(\mu^2) \tag{2.3}$$
$$\zeta_j^{\cdot} = \nu_j \zeta_j + \varkappa_{j-1} \zeta_{j-1} + \mu G_{j1}^* (r, \zeta, t) + O(\mu^2)$$
$$(r = r_1, \ldots, r_{2n+m}; \quad \zeta = \zeta_1, \ldots, \zeta_h; \quad i = 1, \ldots, 2n + m; \quad l = l_1 + \ldots + l_{2n+m})$$

In contrast to the nonresonance case the order of the subsystem corresponding to pure imaginary roots, in system (2.3), is not lowered.

If the equations

$$g_{i-1} r_{i-1} + \mu \sum_{l \geqslant 1} R_{i1}^{(l)} r_1^{l_1} \ldots r_{2n+m}^{l_{2n+m}} = 0$$

have real roots $r_{10}, \ldots, r_{2n+m,0}$, then, by substituting them into the transformations taking (2.1) to (2.3), we obtain the desired stationary solutions which are almost-periodic when the almost-periodic solutions for ζ_j $(j = 1, \ldots, h)$ are determined to within μ from the equations

$$\zeta_j^{\cdot} = \nu_j \zeta_j + \varkappa_{j-1} \zeta_{j-1} + \mu G_{j1}^* (r_0, \zeta, t) \quad (r_0 = r_{10}, \ldots, r_{2n+m,0}) \tag{2.4}$$

The method presented above for the construction of the Liapunov function for the stationary oscillations found under the condition that the roots of the corresponding characteristic equation have negative real parts leads to the inequalities

$$| r_i - r_{i0} | < \mu^{1-\varepsilon}, \quad | \zeta_j - \zeta_{j0} | < \mu^{1-\varepsilon} \tag{2.5}$$

where ε is an arbitrarily small positive number and ζ_{j0} are bounded solutions of (2.4). From inequalities (2.5) it follows that $|x_k - x_{k0}|$ $(k = 1, \ldots, n_1)$ are of order μ for any t in the interval $(0, \infty)$. The latter is not valid in the nonresonance case .

3. As an example let us consider the spatial oscillations of a rigid body in a liquid (see [11], for example), the motion of whose rudders has been approximated by almost-periodic time functions. The origin of the coordinate system $Axyz$ connected with the body is located at the point of application of the buoyancy force, while the coordinate planes coincide with the body's planes of symmetry. The center of mass lies on the axis Ax. The motion of a dynamically and geometrically axisymmetric body in a connected system is described by the following differential equations:

$$u^{\cdot} = \frac{1}{k_{11}'} f_1, \qquad v^{\cdot} = a(\rho_z' f_2 - x_0 f_6), \qquad w^{\cdot} = a(\rho_z' f_3 + x_0 f_5) \tag{3.1}$$

$$p^{\cdot} = \frac{1}{\rho_x'} f_4, \qquad q^{\cdot} = a(x_0 f_3 + k_{22}' f_5), \qquad r^{\cdot} = a(-x_0 f_2 + k_{22}' f_6)$$

$$\varphi^{\cdot} = p - \operatorname{tg}\theta\,(q\cos\varphi - r\sin\varphi), \quad \psi^{\cdot} = \sec\theta\,(q\cos\varphi - r\sin\varphi)$$

$$\theta^{\cdot} = q\sin\varphi + r\cos\varphi$$

$$f_1 = m_0^{-1}(-c_x v_0^2 + c_p\cos\psi\cos\theta) + x_0(q^2 + r^2) + k_{22}'(vr - wq)$$

$$f_2 = m_0^{-1}[c_y v_0 + c_p(\sin\varphi\sin\psi - \cos\varphi\cos\psi\sin\theta)] - k_{11}' ur + k_{22}' wp - x_0 pq$$

$$f_3 = m_0^{-1}[c_z v_0^2 + c_p(\cos\varphi\sin\psi + \sin\varphi\cos\psi\sin\theta)] + k_{11}' uq - k_{22}' vp - x_0 pq$$

$$f_4 = m_0^{-1} b_s m_x v_0^2$$

$$f_5 = m_0^{-1}[m_y v_0^2 - c_g x_c(\sin\psi\cos\varphi + \cos\psi\sin\varphi\sin\theta) - x_0 uq + x_0 vp + P_{zx} pr$$

$$f_6 = m_0^{-1}[m_z v_0^2 + c_g x_c(\sin\psi\sin\varphi - \cos\psi\cos\varphi\sin\theta)] - x_0 ur + x_0 wp - P_{zx} pq$$

$$a = (\rho_z' k_{22}' - x_0^2)^{-1}, \quad P_{zx} = \rho_z' - \rho_x', \quad u = \frac{v_x}{v_s}, \quad v = \frac{v_y}{v_s}$$

$$w = \frac{v_z}{v_s}, \quad p = \omega_x \frac{l}{v_s}, \quad q = \omega_y \frac{l}{v_s}, \quad r = \omega_z \frac{l}{v_s}$$

$$k_{ii}' = 1 + \frac{\lambda_{ii}}{m} \quad (i = 1, 2), \quad k_{44}' = 1 + \frac{\lambda_{44}}{J_x}, \quad k_{66}' = 1 + \frac{\lambda_{66}}{J_z}$$

$$k_{26} = \frac{\lambda_{26}}{ml}, \quad m_0 = \frac{2m}{\rho S l}, \quad \rho_x' = \frac{J_x}{ml^2} k_{44}', \quad \rho_z' = \frac{J_z}{ml^2} k_{66}'$$

$$c_p = \frac{2(G - A)}{\rho S v_s^2}, \quad c_g = \frac{2G}{\rho S v_s^2}, \quad x_c = \frac{x_c^*}{l}, \quad b_s = \frac{b}{l}$$

$$x_0 = x_c + k_{26}, \quad v_0 = \sqrt{u^2 + v^2 + w^2}$$

Here v_x, v_y, v_z are the projections of the velocity of the point of application of the buoyancy force onto the connected axes Ax, Ay, Az , respectively, $\omega_x, \omega_y, \omega_z$ are the projections of the angular velocity, m is the body's mass, J_x and J_z are the axial moments of inertia, λ_{ij} are the coefficients of the joining additional masses, G is the force of gravity, A is the buoyancy force, v_s is the steady-state value of the velocity of translation x_c^* is the coordinate of the center of mass, b

is the coordinate of the point of application of the resultant hydrodynamic forces acting on the rudder, relative to axis Ax, ρ is the liquid's density, S is the characteristic area, l is the characteristic dimension, φ, ψ, θ are the Euler angles. We assume the structure of the hydrodynamic coefficients to be as follows:

$$c_x = c_{x0} + b_3(\alpha^2 + \beta^2), \quad m_x = m_x{}^p p + m_x{}^\varphi \varphi + b_4(\alpha\delta_1 + \beta\delta_2)$$

$$c_y = c_y{}^\alpha \alpha + c_y{}^\delta \delta_2 + c_y{}^r \frac{r}{v_0} + b_1\alpha^3$$

$$m_z = m_z{}^\alpha \alpha + m_z{}^\delta \delta_2 + m_z{}^r \frac{r}{v_0} + c_1\alpha^3$$

$$c_z = -c_y{}^\alpha \beta - c_y{}^\delta \delta_1 - c_y{}^r \frac{q}{v_0} - b_1\beta^3, \quad m_y = m_z{}^\alpha \beta + m_z{}^\delta \delta_1 + m_z{}^r \frac{q}{v_0} + c_1\beta^3$$

$$\alpha = \operatorname{arctg}\left(-\frac{v}{u}\right), \quad \beta = \arcsin\frac{w}{v_0}$$

$$\delta_k = A_k^{(1)} \sin(\omega_k^{(1)} t + \theta_k^{(1)}) + A_k^{(2)} \sin(\omega_k^{(2)} t + \theta_k^{(2)}) \quad (k = 1, 2)$$

Here α and β are the attack and slip angles, δ_k are the rudder deflection angles, $\omega_k{}^{(1)}$ and $\omega_k{}^{(2)}$ are incommensurable frequencies.

We introduce

$$x_0 = \mu x_0{}^*, A_k{}^{(1)} = \mu A_k{}^{*(1)}, \; A_k{}^{(2)} = \mu A_k{}^{*(2)}, \; m_x{}^\varphi = \mu m_x{}^{*\varphi}$$

and we seek the stationary solutions of (3.1) in the form

$$u = u_s + \mu x_1, \; z_i = \mu x_i$$

Here $u_s = (c_p / c_{x_0})^{1/2}$ is the steady-state value of the velocity of the vertical deepening, $z_i = v, w, p, q, r, \varphi, \psi, \theta$ for $i = 2, 3, \ldots, 9$, respectively. Setting $m_z{}^\alpha = -c_y{}^\alpha c g k_{26} / c_p$ and $m_z{}^r = c_y{}^\alpha \rho_z' / k_{22}'$, we obtain, in the characteristic equation of the system for x_i, two pairs of multiple pure imaginary roots with two groups of solutions, three zero roots with three groups of solutions, and two distinct negative roots.

Let us determine the stationary solutions of such a system from the terms of first order in μ. Making transformations analogous to those in Sect. 2 for the resonance case, we obtain

$$\begin{aligned}
&r_i^{\cdot} = \mu R_i + O(\mu^2), \quad (i = 2, 3, 5, \ldots, 9) \\
&\zeta_1^{\cdot} = a_{11}\zeta_1 + \mu G_{11}(r, \zeta_1, \zeta_4, t) + O(\mu^2) \\
&\zeta_4 = a_{44}\zeta_4 + \mu G_{41}(r, \zeta_1, \zeta_4, t) + O(\mu^2) \\
&R_2 = D_{202} r_2, \; R_3 = D_{202} r_3 \\
&R_5 = D_{606} r_5 + D_{609} r_8, \quad R_6 = D_{606} r_6 + D_{609} r_9 \\
&R_7 = D_{707} r_7 + D_{759}(r_5 r_9 - r_6 r_8) \\
&R_8 = -D_{609} r_5 + D_{606} r_8, \qquad R_9 = -D_{909} r_6 + D_{606} r_9
\end{aligned}$$

Here D_{ijk} are known constant coefficients and G_{11} and G_{41} are known polynomials with almost-periodic coefficients. Solving the amplitude equations and finding the stationary solutions for x_i correct within μ, we obtain, for example, for x_2

$$x_2 = -\sum_{\alpha=1}^{2}\left[\frac{M_{20}^{(\alpha)}}{\omega_2^{(\alpha)}} \cos\delta_2^{(\alpha)}\right] + \mu \sum_{\alpha=1}^{2}\left[B(t) H_{204}^{(\alpha)} \sin\delta_1^{(\alpha)} + \right.$$

$$B(t)\, Q_{204}^{(\alpha)} \cos \delta_1^{(\alpha)} + \left(A(t)\, H_{201} + \frac{F_{21}^{(\alpha)}}{\omega_2^{(\alpha)}}\right) \sin \delta_2^{(\alpha)} +$$

$$\left.\left(A(t)\, Q_{201}^{(\alpha)} - \frac{G_{21}^{(\alpha)}}{\omega_2^{(\alpha)}}\right) \cos \delta_2^{(\alpha)}\right]$$

Here $M_{20}^{(\alpha)}, H_{204}^{(\alpha)}, Q_{204}^{(\alpha)}, H_{201}^{(\alpha)}, F_{21}^{(\alpha)}, G_{21}^{(\alpha)}$ are constants, being combinations of the original coefficients, $A(t)$ and $B(t)$ are the solutions of the adjoint system, having the form

$$A(t) = \frac{2u_s x_{10} e^{a_{11} t}}{2u_s + \mu x_{10}(1 - e^{a_{11} t})}, \quad x_{10} = x_1(0)$$

$$B(t) = c_1 \exp\left[a_{44} t + \mu \frac{2a_{44}}{u_s} \int_0^t A(t)\, dt\right]$$

$$(a_{11} < 0,\ a_{44} < 0,\ \delta_1^{(\alpha)} = \omega_1^{(\alpha)} t + \theta_1^{(\alpha)},\ \delta_2^{(\alpha)} = \omega_2^{(\alpha)} t + \theta_2^{(\alpha)})$$

By choosing the hydrodynamic coefficients we can satisfy the existence conditions for the stationary solutions from terms of first order in μ. With prescribed numerical values we have compared the analytic solutions found with the solutions of system (3.1), obtained by numerical methods. The calculation results practically coincide.

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